

## Compositions of Functions

A “composition” can mean many things. It can mean a work of art, music, or writing. It can be a way of combining molecules. It can even be (apparently) a type of doll. In mathematics, it refers to a very specific way of combining functions. There are five main ways to combine two functions to make a new function: add them, subtract them, multiply them, divide them, and *compose* them.

The first four ways are “arithmetic” combinations of functions, which you should have learned about already (although if you haven’t, you can still do this worksheet). The last way, composing functions, is the most complicated, so it gets a section all to itself.

Remember how functions can be viewed as machines? They have inputs, they process the inputs somehow, and they spit out outputs. They’re mathematical machines. Composing two functions is like linking two machines together. It’s taking an input, running it through one machine, taking the *output* of that machine and running that through the second machine. In other words, the output of the first machine becomes the input of the second machine. Like an assembly line. In mathematical notation, we can write it this way:

*Given two functions,  $f$  and  $g$ , the composition  $f \circ g$  is the function defined by  $(f \circ g)(x) = f(g(x))$ .*

1. If a composition of functions means “do one function first and then the other,” in which order do we do  $f \circ g$ ? Does  $f$  come first, or  $g$ ?

2. What’s the difference between  $f \cdot g$  and  $f \circ g$ ?
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3. Let’s start by computing. Let’s say  $f(x) = x^2 - 2$  and  $g(x) = |x + 3|$ . Without computing a formula for  $f \circ g$  or  $g \circ f$ , etc. (because you don’t need to!), compute the following values:

(a)  $(f \circ g)(2) =$

(b)  $(f \circ g)(0) =$

(c)  $(g \circ f)(2) =$

(d)  $(g \circ f)(0) =$

(e)  $(f \circ f)(1) =$

(f)  $(g \circ g)(-5) =$

(g)  $(f \circ g \circ f)(-2) =$

(We didn’t define what the composition of three function is, but take a good guess.)

Of course, sometimes doing things one at a time gets repetitive. Wouldn’t it be nice if we could just come up with a formula for the composition of two functions?

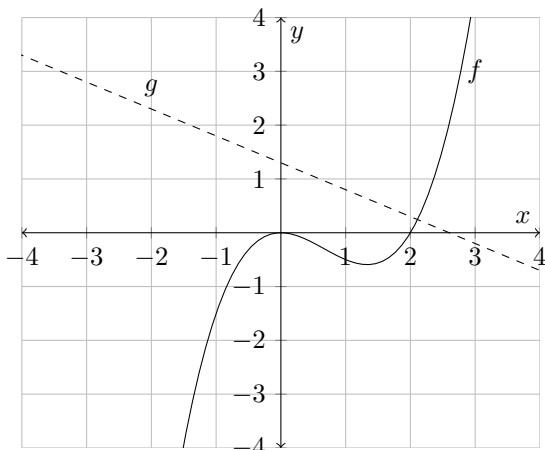
4. If  $f(x) = x^2 - 2$  and  $g(x) = |x + 3|$ , can you:

(a) find a formula for  $f \circ g$ ?

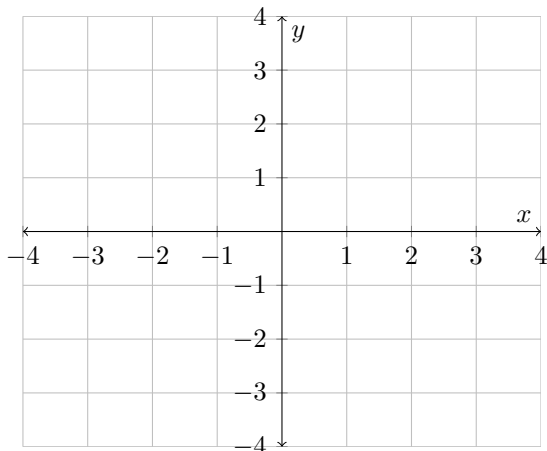
(b) find a formula for  $g \circ f$ ?

Any chance of a graphical interpretation? Let's say we don't have formulas, just graphs.

5. Say  $f$  and  $g$  have the following graphs.



- What is  $(f \circ g)(-2)$ ?
- What is  $(f \circ g)(-1)$ ?
- What is  $(f \circ g)(0)$ ?
- What is  $(f \circ g)(1)$ ?
- What is  $(f \circ g)(2)$ ?
- What is  $(f \circ g)(3)$ ?
- Can you sketch a graph of the function  $f \circ g$ ?



Well done. It's not all that easy to graph a composition of functions, but it can definitely be done (obviously, since you just did it). Let's drive home the fact that you don't need formulas to compute a composition of functions in the next problem.

6. Say you're given some values of  $f$  and  $g$  by the following table.

$x$	-2	-1	0	1	2
$f(x)$	-2	2	0	1	1
$g(x)$	2	2	-1	0	-2

- (a) What is  $(f \circ g)(1)$ ?
- (b) What is  $(f \circ g)(-2)$ ?
- (c) What is  $(f \circ f)(-1)$ ?
- (d) What is  $(g \circ f)(1)$ ?
- (e) What is  $(g \circ g \circ f \circ g)(0)$ ?

We can do this with piecewise functions, too!

7. Let  $f$  and  $g$  be given by

$$f(x) = \begin{cases} 3x^2 & \text{if } x < 0 \\ 2 - x & \text{if } x \geq 0 \end{cases}$$

$$g(x) = \begin{cases} |x - 2| & \text{if } x < 1 \\ 2^x & \text{if } x \geq 1 \end{cases}$$

- (a) What is  $(f \circ g)(1)$ ?
- (b) What is  $(g \circ f)(0)$ ?
- (c) What is  $(f \circ f)(-1)$ ?

Let's play around a bit and explore some properties of compositions of functions. See if the following are true or false, and explain your answers.

8. True or False:  $(f \circ g)(x) = (g \circ f)(x)$ ?
9. True or False:  $(f \circ f)(x) = (f \cdot f)(x)$ ?
10. True or False:  $((f + g) \circ h)(x) = ((f \circ h) + (g \circ h))(x)$ ?

Just for fun, let's try to figure out a rule for the domain of a composition of two functions.

11. Let  $f(x) = \frac{3}{x-1}$  and  $g(x) = \sqrt{x+2}$ .

(a) What is the domain of  $f$ ?

(b) What is the domain of  $g$ ?

(c) Is 1 in the domain of  $f$ ?  $g$ ?  $f \circ g$ ?

(d) Is 0 in the domain of  $f$ ?  $g$ ?  $f \circ g$ ?

(e) Is  $-3$  in the domain of  $f$ ?  $g$ ?  $f \circ g$ ?

(f) In order for  $x$  to be in the domain of  $f \circ g$ , two things must be true. What are they?

1.  $x$  must \_\_\_\_\_.
2. \_\_\_\_\_ must be in the domain of  $f$ .

Many times, we'd rather go in the opposite direction. We might have a complicated function and we want to decompose it into a composition of two functions. For example, if we have the function  $h(x) = (x^3 - 2)^2$ , we could decompose it as  $h = f \circ g$ , where  $f(x) = x^2$  and  $g(x) = x^3 - 2$ . (Check that to be sure you agree!) Can you do the same? This might involve some trial and error, especially at first.

12. Write  $h$  as the composition of two functions,  $f$  and  $g$ , such that  $h = f \circ g$ .

$$h(x) = \frac{3}{x^4 - 5}$$

13. Write  $h$  as the composition of two functions,  $f$  and  $g$ , such that  $h = f \circ g$ .

$$h(x) = e^{x-5}$$

14. Write  $h$  as the composition of two functions,  $f$  and  $g$ , such that  $h = f \circ g$ .

$$h(x) = |x^2| + 4x^2$$

15. Write  $h$  as the composition of two functions,  $f$  and  $g$ , such that  $h = f \circ g$ .

$$h(x) = \frac{1}{x} + \frac{2}{x^2}$$

16. Write  $h$  as the composition of two functions,  $f$  and  $g$ , such that  $h = f \circ g$ .

$$h(x) = \frac{3x^2}{x^4 - 5}$$